SPECIAL STANDARD STATIC SPACE-TIMES [1]

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Abstract

Essentially, some conditions for the Riemannian factor and the warping function of a standard static space-time are obtained in order to guarantee that no nontrivial warping function on the Riemannian factor can make the standard static space-time Einstein.

1   Introduction

In order to obtain general solutions to Einstein's field equations, Lorentzian warped product manifolds were introduced in general relativity (see [7, 30]). Generalized Robertson-Walker space-times and standard static space-times are two well known important examples. The former are clearly a generalization of Robertson-Walker space-times and the latter a generalization of the Einstein static universe. In this paper, we basically focus on properties of the Ricci tensor and scalar curvature of a standard static space-time.

We recall the definition of a warped product of two pseudo-Riemannian manifolds (B, g_B) and (F, g_F) with a smooth function $b \colon B \to (0, \infty)$ (see [7, 30]). Suppose that (B, g_B) and (F, g_F) are pseudo-Riemannian manifolds and also suppose that $b \colon B \to (0, \infty)$ is a smooth function. Then the (singly) warped product, $B \times_b F$ is the product manifold $B \times F$ equipped with the metric tensor $g = g_B \oplus b^2 g_F$ defined by

$$g = \pi^*(g_B) \oplus (b \circ \pi)^2 \sigma^*(g_F)$$

where $\pi \colon B \times F \to B$ and $\sigma \colon B \times F \to F$ are the usual projection maps and * denotes the pull-back operator on tensors. Here, (B, g_F) is called as the

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base manifold and (F, g_F) is called as the fiber manifold and also b is called as the warping function. There are also different generalizations of warped products such as warped products with more than one fiber manifold, called multiply warped products (see [35]) or warped products with two warping functions acting symmetrically on the fiber and base manifolds, called doubly warped products (see [34]). Finally, a warped product is said to be a twisted product if the warping function defined on the product of the base and fiber manifolds (see [16]).

Basically, a standard static space-time can be considered as a Lorentzian warped product where the warping function is defined on a Riemannian manifold and acting on the negative definite metric on an open interval of real numbers. More explicitly, a standard static space-time, $_f(a, b) \times F$ is a Lorentzian warped product furnished with the metric $g = -f^2 \mathrm{d}t^2 \oplus g_F$, where (F, g_F) is a Riemannian manifold, $f \colon F \to (0, \infty)$ is smooth, and $-\infty \leq a < b \leq \infty$. This class of space-times has been previously considered by many authors. Now, we give a summary of some major work about standard static space-times. In [30], it was shown that any static space-time is locally isometric to a standard static space-time. Kobayashi and Obata [26] stated the geodesic equation for this class of space-times and the causal structure and geodesic completeness was considered in [3], where sufficient conditions on the warping function for nonspacelike geodesic completeness of the standard static space-time was obtained (see also [32] and [34]). In [2], conditions are found which guarantee that standard static space-times either satisfy or else fail to satisfy certain curvature conditions from general relativity. The existence of geodesics in standard static space-times have been studied by several authors. Sánchez [33] gives a good overview of geodesic connectedness in semi-Riemannian manifolds, including a discussion for standard static space-times. In [4], geodesic structure of standard static space-times is studied and conditions are found on the warping function to imply non-returning and pseudo-convex geodesic systems on a standard static space-time.

The Minkowski space-time and the Einstein static universe are two most famous examples of standard static space-times (see [7, 19]) which is $\mathbb{R} \times \mathbb{S}^3$ equipped with the metric

$$g = -\mathrm{d}t^2 + (\mathrm{d}r^2 + \sin^2 r \mathrm{d}\theta^2 + \sin^2 r \sin^2 \theta \mathrm{d}\phi^2)$$

where $\mathbb{S}^3$ is the usual 3-dimensional Euclidean sphere and the warping function $f \equiv 1$. Another well-known example is the universal covering space of anti-de Sitter space-time, a standard static space-time of the form $_f\mathbb{R} \times \mathbb{H}^3$

where $\mathbb{H}^3$ is the 3-dimensional hyperbolic space with constant negative sectional curvature and the warping function $f\colon \mathbb{H}^3 \to (0,\infty)$ defined as $f(r,\theta,\phi) = \cosh r$ (see [7, 19]). As a final example, we can also mention the Exterior Schwarzschild space-time (see [7, 19]), a standard static space-time of the form $_f\mathbb{R} \times (2m,\infty) \times \mathbb{S}^2$, where $\mathbb{S}^2$ is the 2-dimensional Euclidean sphere, the warping function $f\colon (2m,\infty) \times \mathbb{S}^2 \to (0,\infty)$ is given by $f(r,\theta,\phi) = \sqrt{1 - 2m/r}$, $r > 2m$ and the line element on $(2m,\infty) \times \mathbb{S}^2$ is

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2).$$

In literature, it is not known in full generality that whether there exists a nontrivial warping function f for which the warped product pseudo-Riemannian (or Riemannian) manifold $_fB \times F$ is Einstein for given pseudo-Riemannian (or Riemannian) manifolds (F, g_F) and (B, g_B). In fact, the answer of this question depends on whether there exists a nontrivial common solution f to some differential equations on (F, g_F). (See page 267 of [8]). This problem was considered especially for Einstein Riemannian warped products with compact base and some partial answers were also provided (see [18, 22, 23, 24]). In [23], it is proven that an Einstein Riemannian warped product with a non-positive scalar curvature and compact base is just a trivial Riemannian product. Constant scalar curvature of warped products was studied in [10, 12, 14, 15] when the base is compact and of generalized Robertson-Walker space-times in [14]. Furthermore, partial results for warped products with non-compact base were obtained in [6] and [9]. The physical motivation of existence of a positive scalar curvature comes from the positive mass problem. More explicitly, in general relativity the positive mass problem is closely related to the existence of a positive scalar curvature (see [37]). As a more general related reference, one can consider [21] to see a survey on scalar curvature of Riemannian manifolds.

The problem of existence of a warping function which makes the warped product Einstein was already studied for special cases such as generalized Robertson-Walker space-times and a table given summarizing different cases of Einstein Ricci tensor of a generalized Robertson-Walker when the Ricci tensor of the fiber is Einstein in [1] (see also references therein). In this paper, we consider this problem for standard static space-times. In fact, we essentially investigate the conditions for (F, g_F) so that there exists no nontrivial function f on F guaranteing that the standard static space-time $_f(a,b) \times F$ is Einstein. Although the results in this paper remain valid for the Riemannian setting, that is, when (B, g_B) is the Euclidean interval $((a,b), \mathrm{d}t^2)$, we prefer to state these results in Lorentzian setting since there

are certain standard static space-times in Lorentzian geometry, such as, Einstein static universe, Schwarzschild exterior space-time and (universal) anti-de Sitter space-time, which are of interest in Relativity Theory and appear as examples to the results of this paper. See *Remark 4.11* for the summary of the results of this paper from the view of this Introduction.

Einstein Ricci tensor and constant scalar curvature of standard static space-times with perfect fluid were already considered in (see [26, 29]). Note that a matter is called a perfect fluid if the energy-momentum tensor T has the form

$$T = (\mu + p)W \otimes W + pg,$$

where W is a 1-form with $g(W, W) = -1$ and μ and p are called the energy density and the pressure, respectively. In [26], it is shown that a standard static space-time $_f(a, b) \times F$ has a perfect fluid if

$$\mathrm{Ric}_F - \frac{\tau_F}{s}g_F = \frac{1}{f}\Big(\mathrm{H}_F^f - \frac{\Delta_F(f)}{s}g_F\Big),$$

where Ric is the Ricci tensor, H is the Hessian form and τ_F is the scalar curvature on (F, g_F) and also $\dim(F) = s$ (see also [19, 30, 29]). Moreover, in [27], the conformal tensor on standard static space-times with perfect fluid is studied and it is shown that a standard static space-time with perfect fluid is conformally flat if and only if its fiber is Einstein and hence of constant curvature.

In this paper, we will consider arbitrary standard static space-times, i.e., we will not assume the existence of a perfect fluid, our results can be considered as extensions of the results in [26, 27, 29] where standard static space-times with perfect fluid were considered. Duggal studied the scalar curvature of 4-dimensional triple Lorentzian products of the form $L \times B \times_f F$ and obtained explicit solutions for the warping function f to have a constant scalar curvature for this class of products (see [13]). We also discuss conditions on the warping function or on the fiber of a standard static space-time to have a constant scalar curvature on the space-time. Especially, we show that the Einstein static universe cannot be generalized to a standard static space-time $_f\mathbb{R} \times \mathbb{S}^s$ modelling a "static universe" with a nonconstant warping function f on $(S^s, d\sigma^2)$, that is, to a standard static space-time with constant scalar curvature in *Theorem 3.1*.

2 Preliminaries

In this section, we give the formal definition of a standard static space-time and state some curvature formulas (see [7, 30]).

Definition 2.1. Let (F, g_F) be an s-dimensional Riemannian manifold and $f \colon F \to (0, \infty)$ be a smooth function. Then $n(= s + 1)$-dimensional product manifold $(a, b) \times F$ furnished with the metric tensor $g = -f^2 \mathrm{d}t^2 \oplus g_F$ is called a standard static space-time and is denoted by $_f(a, b) \times F$, where $\mathrm{d}t^2$ is the Euclidean metric tensor on (a, b) and $-\infty \leq a < b \leq \infty$.

Throughout the paper the fiber (F, g_F) of a standard static space-time of the form $_f(a, b) \times F$ is always assumed to be connected. Now we state some curvature formulas for standard static space-times to be used later in the proofs. Note that, since $_f(a, b) \times F$ and $F \times_f (a, b)$ are isometric, the curvature formulas below can easily be obtained from the well known curvature formulas for warped product metric tensors by making suitable substitutions (see for example, [7, 8, 30, 34]).

Here, we use sign convention for the Laplacian in [30], i.e., defined by or $\Delta = \mathrm{tr}(\mathrm{H})$, (see page 86 of [30]) where H is the Hessian form (see page 86 of [30]) and tr denotes for the trace, or equivalently, $\Delta = \mathrm{div}(\mathrm{grad})$, where div is the divergence and grad is the gradient (see page 85 of [30]).

Proposition 2.2. *Let $_f(a, b) \times F$ be a standard static space-time. If τ and τ_F denote the scalar curvatures of the space-time and fiber, respectively, then*

$$\tau = \tau_F - 2\frac{\Delta_F(f)}{f},$$

where Δ_F denotes the Laplace operator on F.

Remark 2.3. Note that, in the above Proposition, since $\tau(t, p)$ is independent of $t \in (a, b)$, τ is a lift of a unique function $\tilde{\tau}$ on F to $(a, b) \times F$. For brevity in expressions, sometimes we abuse the notation and write τ instead of $\tilde{\tau}$ on F to avoid taking the lifts of functions on F to $(a, b) \times F$.

Proposition 2.4. *Let $_f(a, b) \times F$ be a standard static space-time and also let V and W be vector fields on F. If Ric and Ric_F denote the Ricci tensors of the space-time and the fiber, respectively, then*

$$\mathrm{Ric}(\frac{\partial}{\partial t} + V, \frac{\partial}{\partial t} + W) = \mathrm{Ric}_F(V, W) + f\Delta_F(f) - \frac{1}{f}\mathrm{H}_F^f(V, W),$$

where H_F^f is the Hessian form of f on (F, g_F).

3 Standard Static Space-times of Constant Scalar Curvature

In this Section, we essentially investigate geometric and topological conditions on a Riemannian manifold (F, g_F) which yield the constancy of the warping function f on F of a constant scalar curvature standard static space-time $_f(a, b) \times F$.

Theorem 3.1. *Let $_f(a, b) \times F$ be a standard static space-time where $s \geq 2$. Assume that (F, g_F) is compact and the scalar curvature τ of the space-time is constant. Then, (F, g_F) is of constant scalar curvature τ_F if and only if f is constant on F. In either case, $\tau = \tau_F$.*

Proof. First note that, $\int_F (\tau_F - \tau) f = 0$ by *Proposition 2.2* and page 104 of [5]. Thus, since $f > 0$ on M, $\tau_F(p_0) = \tau$ at some $p_0 \in F$. Now, if τ_F is constant on F then $\tau_F(p) = \tau$ for all $p \in F$, and it follows from *Proposition 2.2* that $\Delta_F(f) = 0$ on (F, g_F). Thus, f is constant on F. Conversely, if f is constant on F then, by *Proposition 2.2*, $\tau_F(p) = \tau$ for all $p \in F$, and hence τ_F is constant on F. □

Note that, one of the ingredients of the physical concept of a "static universe" (see [17]) is the constancy of the scalar curvature of the space-time modelling a "static universe", which in fact, corresponds to the constancy of the trace of the stress-energy tensor of the space-time via Einstein equation. Note that, the Einstein static universe $\mathbb{R} \times \mathbb{S}^s$ is of constant scalar curvature, where $\mathbb{S}^s = (S^s, d\sigma^2)$ is the unit Euclidean sphere. (See page 189 of [7]). Thus, by *Theorem 3.1*, the Einstein static universe cannot be generalized to a standard static space-time $_f\mathbb{R} \times \mathbb{S}^s$ modelling a "static universe" with a nonconstant warping function f on $(S^s, d\sigma^2)$, that is, to a standard static space-time with constant scalar curvature.

It is noticed that one cannot obtain a non-trivial standard static space-time of a constant scalar curvature when it has a compact fiber of a constant scalar curvature. Thus we should focus on standard static space-times with compact fibers of nonconstant scalar curvatures. In [15], a similar problem was considered on a wider class of warped products (see also [14]). In order to make use of [15], we introduce a linear operator $L : H^{1,2}(F) \rightarrow H^{1,2}(F)$ on a compact Riemannian manifold (F, g_F) defined by

$$L(v) = -\Delta_F(v) + \frac{\tau_F(q)}{2} v,$$

where v in the Sobolev space $H^{1,2}(F)$. Then we are ready to state the following result.

Theorem 3.2. *Let (F, g_F) be a compact Riemannian manifold with variable scalar curvature $\tau_F \colon F \to \mathbb{R}$ where $s \geq 2$. Then there exists a smooth function $f \colon F \to (0, \infty)$ such that the corresponding standard static space-time $_f(a, b) \times F$ is of constant scalar curvature τ.*

Proof. From *Proposition 2.2*, like in [12] and [15], we look for $\tau \in \mathbb{R}$ and $f \in \mathcal{C}^\infty(F)$ such that $\mathrm{L}f = \tau f$. It is well known that this type of eigenvalue problem has only one eigenvalue, $\lambda_1(\tau_F)$, (which is simple) such that the corresponding eigenfunction is strictly positive (see [5]). Note the centrality of the compactness of F. □

Remark 3.3. The constant scalar curvature of the standard static space-time $\tau = 2\lambda_1$ where λ_1 is the first eigenvalue of the operator L on $\mathrm{H}^{1,2}(F)$.

Thus it is possible to produce a non-trivial standard static space-time of constant scalar curvature when the fiber is compact and has a nonconstant scalar curvature. Now we turn our attention to the complete case (not necessarily compact), roughly speaking, under suitable hypothesis for the curvature of the fiber, we will give a necessary condition for constant scalar curvature in a standard static space-time.

Theorem 3.4. *Let (F, g_F) be a complete manifold without boundary where $s \geq 2$. Suppose the Ricci curvature of F is non-negative, and suppose $\Delta_F(\tau_F) \leq 0$ and also $\|\nabla_F(\tau_F)\| = o(r(x))$, where $r(x)$ is the distance from x to some fixed point $p \in F$. If $_f(a, b) \times F$ is a standard static space-time with constant scalar curvature τ, then $\tau \leq \inf_F(\tau_F)$.*

Proof. By contradiction, suppose that $\tau > \inf_F(\tau_F)$. *Proposition 2.2* implies that
$$\Delta_F(f) - \left(\frac{\tau_F}{2} - \frac{\tau}{2}\right)f = 0,$$
with τ constant and f is positive. Let $q = (\tau_F - \tau)/2$, since $\tau > \inf_F(\tau_F)$ there results $\inf_F(q) < 0$. thus by Corollary 1.1 of [28], we obtain a contradiction. □

Remark 3.5. In the previous theorem, we may also require (F, g_F) be compact and in this case, by the variational structure of the principal eigenvalue of the operator $-\Delta_F + \tau_F/2$, namely
$$\frac{\tau}{2} = \inf_{\mathrm{H}^{1,2}(F)} \frac{\int_F \|\nabla_F(u)\|^2 + \frac{\tau_F}{2}u^2}{\int_F u^2},$$

there results

$$\frac{\tau}{2} \geq \inf_{\mathrm{H}^{1,2}(F)} \frac{\int_F \|\nabla_F(u)\|^2 + \frac{\inf_F(\tau_F)}{2}u^2}{\int_F u^2} = \frac{\inf_F(\tau_F)}{2}.$$

Here, notice that $\inf_{\mathrm{H}^{1,2}(F)} \frac{\int_F \|\nabla_F(u)\|^2 + \frac{\inf_F(\tau_F)}{2}u^2}{\int_F u^2}$ is the principal eigenvalue of $-\Delta_F + \inf_F(\tau_F)/2$ on the compact manifold F. So, $\tau \geq \inf_F(\tau_F)$. But by the above theorem $\tau \leq \inf_F(\tau_F)$, thus $\tau = \inf_F(\tau_F) \geq 0$, since the latter inequality holds because of the non-negative Ricci curvature of F.

We now state a simple result for 2-dimensional standard static space-times with constant scalar curvatures. Note that if $M =_f (a,b) \times (c,d)$ is a 2-dimensional standard static space-time with the metric tensor $g = -f^2 \mathrm{d}t^2 + \mathrm{d}x^2$, then its scalar curvature τ is given by $\tau = -2f''/f$, where $-\infty \leq a < b \leq \infty$ and $-\infty \leq c < d \leq \infty$ also $f\colon (c,d) \to (0,\infty)$ is smooth.

Proposition 3.6. *Let $_f(a,b) \times (c,d)$ be a 2-dimensional standard static space-time. Then the scalar curvature τ is constant if f satisfies one of the followings*

1. *$f(x) = c_1 x + c_2$, for some $c_1, c_2 \in \mathbb{R}$ when $\tau = 0$,*

2. *$f(x) = c_1 \exp(\sqrt{-2\tau}x) + c_2 \exp(-\sqrt{-2\tau}x)$, for some $c_1, c_2 \in \mathbb{R}$ when $\tau < 0$,*

3. *$f(x) = c_1 \cos(\sqrt{2\tau}x) + c_2 \sin(\sqrt{2\tau}x)$, for some $c_1, c_2 \in \mathbb{R}$ when $\tau > 0$.*

One can compare the previous result with the characterization of constant Gauss curvature revolution of surfaces embedded in $\mathbb{R}^3$ (see the examples in page 66-67 of [25], chapter 3 of [31] and page 169 of [11]).

4   Einstein Standard Static Space-times

We now concentrate on the Ricci tensor of standard static space-times. More precisely, we will try to determine conditions on the warping function of a standard static space-time so that the space-time becomes Einstein or Ricci-flat when the fiber is Einstein or Ricci-flat.

Recall that an arbitrary n-dimensional pseudo-Riemannian manifold (M, g) is said to be Einstein with λ if there exists a smooth map $\lambda\colon M \to \mathbb{R}$ such that $\mathrm{Ric} = \lambda g$. Furthermore, if (M, g) is Einstein with λ and $\dim(M) = n \geq 3$, then λ is constant and $\lambda = \tau/n$, where τ is the (constant) scalar curvature of (M, g). Also note that for a 2-dimensional Einstein manifold (M, g) with λ, one cannot necessarily conclude the constancy of λ (see [30]).

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Proposition 4.1. *Let (F, g_F) be an s-dimensional Riemannian manifold with scalar curvature τ_F where $s \geq 2$ and let the standard static space-time $_f(a, b) \times F$ be Einstein with constant scalar curvature τ. Then,*

a) $\Delta_F(f) = -\frac{\tau}{n} f$ and $\mathrm{Ric}_F = \frac{1}{f} \mathrm{H}_F^f + \frac{\tau}{n} g_F$ on (F, g_F), where $n = s + 1$.

b) $(s + 1)\tau_F = (s - 1)\tau$.

Proof. By assumption, since $\mathrm{Ric} = \frac{\tau}{n}(-f^2 dt^2 \oplus g_F)$, we obtain from *Proposition 2.4* that,

$$\mathrm{Ric}_F(V, W) + f\Delta_F(f) - \frac{1}{f}\mathrm{H}_F^f(V, W) = -\frac{\tau}{n}f^2 + \frac{\tau}{n}g_F(V, W)$$

for all $V, W \in \Gamma TF$.

a) If we set $V = 0 = W$ in the above expression, we obtain $\Delta_F(f) = -\frac{\tau}{n}f$ and hence, it also follows that $\mathrm{Ric}_F(V, W) = \frac{1}{f}\mathrm{H}_F^f(V, W) + \frac{\tau}{n}g_F(V, W)$ for all $V, W \in \Gamma TF$.

b) Note that, if we take the trace of the equation $\mathrm{Ric}_F = \frac{1}{f}\mathrm{H}_F^f + \frac{\tau}{n}g_F$ with respect to g_F on F, we obtain $\tau_F = \frac{1}{f}\Delta_F(f) + \frac{s}{n}\tau$, and hence, it follows from *(a)* that $(s + 1)\tau_F = (s - 1)\tau$.

□

Remark 4.2. Let (F, g_F) be a Riemannian manifold of scalar curvature τ_F where $s \geq 2$ and let the standard static space-time $_f(a, b) \times F$ be Einstein with constant scalar curvature τ. Then, by *Theorem 3.1* and *Proposition 4.1*, we conclude the following:

a) (F, g_F) is of constant scalar curvature τ_F.

b) If (F, g_F) is compact then $\tau = 0$, $\tau_F = 0$ and f is constant on F.

Remark 4.3. Note that, in *Proposition 4.1*, if we further assume that (F, g_F) is Einstein with scalar curvature τ_F then, since $\mathrm{Ric}_F = \frac{\tau_F}{s}g_F$ on (F, g_F), we obtain by using *Proposition 4.1* that $\mathrm{H}_F^f = (\frac{\tau_F}{s} - \frac{\tau}{n})fg_F = -\frac{\tau_F}{s(s-1)}g_F = -\frac{\tau}{s(s+1)}g_F$ on (F, g_F). (Note that, by *Proposition 4.1-b*, τ_F is also constant when $\dim F = s = 2$).

Theorem 4.4. *Let (F, g_F) be a complete Riemannian manifold with non-negative Ricci curvature where $s \geq 2$. If the standard static space-time $_f(a, b) \times F$ is Ricci flat then f is constant on F.*

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Proof. Since $_f(a, b) \times F$ is Einstein with scalar curvature $\tau = 0$, it follows from *Proposition 4.1* that $\Delta_F(f) = 0$ on (F, g_F). Since f is positive on F, f is constant by *Corollary 1* of [36]. $\square$

Note that, by *Theorem 4.4*, if a standard static space-time $_f(a, b) \times F$ is Ricci flat with a nonconstant warping function f on F then (F, g_F) is either incomplete or not of nonnegative Ricci curvature (or both). Hence, for the Schwarzschild exterior space-time $_f\mathbb{R} \times F$ (see page 367 of [30]), we conclude that (F, g_F) is either incomplete or not of nonnegative Ricci curvature, where $N = (2m, \infty) \times S^2$ and $g_F = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + d\sigma^2$. Indeed, for the Schwarzschild exterior space-time, (F, g_F) is both incomplete and not of nonnegative Ricci curvature.

Remark 4.5. Note that the converse of *Theorem 4.4* is not true in general. For example, the $n(\geq 3)$-dimensional Einstein static universe is an counterexample to this case.

Theorem 4.6. *Let (F, g_F) be a complete Einstein $s(\geq 2)$-dimensional Riemannian manifold with scalar curvature τ_F. If the standard static space-time $_f(a, b) \times F$ is Einstein with scalar curvature τ then $\tau \leq 0$. Furthermore,*

  a) if $\tau = 0$ then f is constant on F,

  b) if $\tau < 0$ then f is nonconstant on F and (F, g_F) is a warped product of the Euclidean line and a complete Riemannian manifold with warping function ψ on the real line satisfying the equation $\frac{d^2\psi}{dt^2} + \frac{\tau}{s(s+1)}\psi = 0$, $\psi > 0$.

Proof. Suppose that $\tau > 0$. Then by *Proposition 4.1*, $\tau_F = \frac{s-1}{s+1}\tau > 0$ and it follows from Myers theorem that (F, g_F) is compact. Hence, by *Theorem 3.1*, f is constant on F. But this conflicts with $\Delta_F(f) = -\frac{\tau}{n}f$ since $f > 0$ on F (see *Proposition 4.1*). Thus $\tau \leq 0$. Furthermore,

  *a) if $\tau = 0$ then, by *Proposition 4.1*, $\Delta_F(f) = 0$ on (F, g_F) and $\text{Ric}_F = \frac{\tau_F}{s}g_F = 0$ since $\tau_F = 0$. Thus, since f is positive on F, f is constant by *Corollary 1* of [36].*

  *b) if $\tau < 0$ then, by *Proposition 4.1* and *Remark 4.3*, f is nonconstant and $\text{H}_F^f = -\frac{\tau_F}{s(s-1)}g_F$ on (F, g_F), where $\tau_F = \frac{s-1}{s+1}\tau < 0$. Hence, it follows from *Corollary E* of [20] that (F, g_F) is a warped product of the Euclidean line and a complete Riemannian manifold with warping function ψ on the real line satisfying the equation $\frac{d^2\psi}{dt^2} + \frac{\tau}{s(s+1)}\psi = 0$, $\psi > 0$.*

$\square$

Note that, the (universal) anti-de Sitter space-time $_f\mathbb{R} \times \mathbb{H}^s$ of constant sectional curvature -1 is an example to *Theorem 4.6-b* for $s \geq 2$. (See page 183 of [7]). Indeed, note that the Riemannian hyperbolic space $\mathbb{H}^s$ can be written as a warped product of the Euclidean line and the Euclidean space with warping function $\psi = e^{\pm t}$ on the real line.

As we did in *Section 3* we will consider the same type of problems on a standard static space-time by using [28] when the fiber is complete without boundary.

Theorem 4.7. *Let (F, g_F) be a complete manifold without boundary where $s \geq 2$. Suppose the Ricci curvature of F is non-negative, and suppose $\Delta_F(\tau_F) \leq 0$ and also $\|\nabla_F(\tau_F)\| = o(r(x))$, where $r(x)$ is the distance from x to some fixed point $p \in F$. If $_f(a,b) \times F$ is an Einstein standard static space-time then $\tau \leq \frac{s+1}{s} \inf_F(\tau_F)$.*

Proof. First note that the space-time is Einstein this means that Ric $= \lambda g$, where $(s+1)\lambda = \tau$ and τ is constant. Considering the trace in (F, g_F) in *Proposition 4.1(a)*, there results a positive solution f for the Schrödinger equation $-\Delta_F(f) + q_E(f)$, where $q_E = \tau_F - \frac{s}{s+1}\tau$. On the other hand, q_E verifies $\Delta_F(q_E) \leq 0$ and $\|\nabla_F(q_E)\| = o(r(x))$ with $r(x)$ as in the hypothesis. So by Corollary 1.1 of [28], $0 \geq \inf_F(q_E) = \inf_F - \frac{s}{s+1}\tau$, or equivalently $\tau \leq \frac{s+1}{s} \inf_F(\tau_F)$. $\square$

Theorem 4.8. *Let (F, g_F) be a complete manifold without boundary where $s \geq 2$. Suppose the Ricci curvature of F is non-negative. If $_f(a,b) \times F$ is an Einstein standard static space-time then $\tau = \tau_F = 0$ and hence the space-time is Ricci-flat.*

Proof. As again in the proof of the previous theorem, note that the space-time is Einstein this means that Ric $= \lambda g$, where $(s+1)\lambda = \tau$ and τ is constant. By *Proposition 4.1(a)*, $\tau_F = \frac{s-1}{s+1}\tau$ is constant. Thus the hypothesis of *Theorem 4.7* are satisfied and this leads $\tau \leq \frac{s+1}{s} \inf_F(\tau_F) = \frac{s-1}{s}\tau$. On the other hand, as Ricci curvature of F is non-negative, $\tau = \frac{s+1}{s-1}\tau_F \geq 0$. So, $\tau = 0$. $\square$

Corollary 4.9. *Let (F, g_F) be a complete manifold without boundary where $s \geq 2$. Suppose the Ricci curvature of F is non-negative. If $_f(a,b) \times F$ is an Einstein standard static space-time then f is constant and (F, g_F) is Ricci-flat.*

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Proof. It just follows *Theorem 4.4* and *Theorem 4.8* and also *Proposition 4.1(a)*. □

As a consequence, we obtain the result that follows:

Theorem 4.10. *Let (F, g_F) be a complete manifold without boundary where $s \geq 2$. If $_f(a, b) \times F$ is an Einstein standard static space-time then either (F, g_F) is Ricci-flat or the Ricci curvature of F cannot be non-negative.*

In the remark below, we collect the results in this section from viewpoint of the nonexistence of a nonconstant warping function f on a connected, complete Riemannian manifold (F, g_F) for which the standard static space-time $_f(a, b) \times F$ is Einstein.

Remark 4.11. Let (F, g_F) be a Riemannian manifold and $_f(a, b) \times F$ be a standard static space-time where $s \geq 2$.

a) If (F, g_F) is compact and of constant scalar curvature (or Einstein) then there exists no nonconstant function f on F for which $_f(a, b) \times F$ is of constant scalar curvature (or Einstein). (See *Theorem 3.1*).

b) If (F, g_F) is complete and of nonnegative Ricci curvature then there exists no nonconstant function f on F for which $_f(a, b) \times F$ is Ricci flat. (See *Theorem 4.4*).

c) If (F, g_F) is complete and Einstein then there exists no nonconstant function f on F for which $_f(a, b) \times F$ is Ricci flat. (See *Theorem 4.6*).

d) If (F, g_F) is complete and of non-negative Ricci curvature, then there exists no nonconstant function f on F for which $_f(a, b) \times F$ is Einstein. (See *Corollary 4.9*).

Here note that, the above Remark remains valid (as well as other results in this section) in the Riemannian setting, that is, when we take (a, b) as an (Euclidean) interval with metric tensor $\mathrm{d}t^2$.

We now consider 2-dimensional Einstein standard static space-times. Let $M =_f (a, b) \times (c, d)$ be a 2-dimensional standard static space-time with the metric tensor $g = -f^2 \mathrm{d}t^2 + \mathrm{d}x^2$. If $\frac{\partial}{\partial t} \in \mathfrak{X}(a, b)$ and $\frac{\partial}{\partial x} \in \mathfrak{X}(c, d)$, then

$$\mathrm{Ric}(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}) = f f'' - \frac{f''}{f},$$

where $-\infty \leq a < b \leq \infty$ and $-\infty \leq c < d \leq \infty$ also $f : (c, d) \to (0, \infty)$ is smooth. Thus we can easily state the following result.

Proposition 4.12. *Let $_f(a,b) \times (c,d)$ be a 2-dimensional standard static space-time. Then*

1. *the space-time is Einstein with λ if and only if $f'' = -\lambda f$,*

2. *the space-time is Ricci-flat if and only if $f(x) = c_1 x + c_2$ on (c,d), for some $c_1, c_2 \in \mathbb{R}$.*

In the previous result, since we cannot conclude the constancy of λ, it is impossible for us to obtain explicit solutions. However, we can only say that $\lambda \colon (a,b) \times (c,d) \to \mathbb{R}$ depends only on the second variable, i.e., for each $x \in (c,d)$, we have $\lambda(t_1, x) = \lambda(t_2, x)$ for any $t_1, t_2 \in (a,b)$.

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